Lecture 2: Improper integrals part 2, area between 2 curves

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Ex)

$$\int_0^6 \frac{1}{x}\,\mathrm{d}x$$ x cant equal 0, so sub in t

$$\sim \int_t^6 \frac{1}{x}\,\mathrm{d}x \Big]_t^6$$

$= \ln(x)$

$=\lim_{t\to 0}(\ln 6 - \ln t) \to -\infty$

$= \ln 6 - \lim_{t\to 0} \ln t$

$= \ln 6 - (-\infty)$

$= \infty$ area is infinite!

This is one type of an improper integral.

We call integrals $\int_a^b f(x)\,\mathrm{d}x = \infty$ **divergent** (the area under the curve is infinite).

If $\int_a^b f(x)\,\mathrm{d}x$ is finite despite *f(x)* being discontinuous on [a,b] then we call the integral **convergent**.

Another type of improper integral is when one or more bounds are $\pm\infty$.

$$\int_a^b f(x)\,\mathrm{d}x, \qquad a \text{ and/or b equals } \pm\infty$$

Ex)

$$\int_1^\infty \frac{1}{x}dx$$

Again, introduce the variable t, and take the limit as t approaches ∞.

$$\lim_{t\to\infty} \int_1^t \frac{1}{x}dx \Big]_1^t$$

0

$= \lim_{t\to\infty} \ln x$

$= \lim_{t\to\infty} (\ln t - \ln 1)$

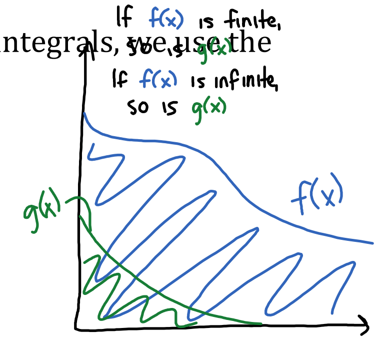
$= \lim_{t\to\infty} \ln t$

$= \infty$

So, $\int_1^t \frac{1}{x}dx$ diverges.

Sometimes, you'll have an integral that you can't solve immediately with the methods we know, such as $\int_0^{\frac{\pi}{2}} \frac{1}{\sin x}\, dx$.

To determine convergence/divergence of these integrals, we use the <u>Comparison Theorem</u>:



$f(x) \geq g(x) \geq 0$

a) If $\int_a^b f(x)\,\mathrm{d}x$ converges, so does $\int_a^b g(x)\,\mathrm{d}x$.

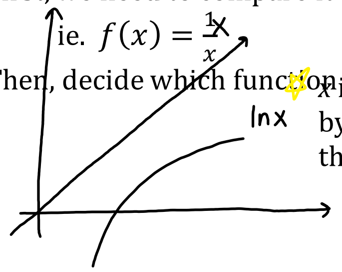
b) If $\int_a^b f(x)\,\mathrm{d}x$ diverges, so does $\int_a^b g(x)\,\mathrm{d}x$.

<u>Ex</u>) Is $\int_2^{\infty} \frac{1}{\ln(x)}\, dx$, divergent or convergent?

First, we need to compare it to an easier function that we can actually integrate,

ie. $f(x) = \frac{1}{x}$

Then, decide which function is larger (sketch).



x is bigger than ln x; so dividing 1 by *ln* x results in a larger fraction than dividing 1 by x.

true for both the functions and the integrals

$$\frac{1}{\ln x} \geq \frac{1}{x}, \qquad x \geq 2$$

So, if we can show that $\int_2^{\infty} \frac{1}{x}\, dx$ diverges, then $\int_2^{\infty} \frac{1}{\ln(x)}\, dx$ diverges too.

$$\lim_{t\to\infty} \int_2^t \frac{1}{x}\,\mathrm{d}x$$
$$= \lim_{t\to\infty} (\ln(t) - \ln(2))$$
$$= \infty$$

improper because $x \neq 0$

<u>Ex</u>)

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}\, dx$$

$u = \frac{1}{x}$

$du = \frac{-1}{x^2}\, dx$

$dx = -x^2\, du$

$x = \frac{1}{u}$

$$= \lim_{t\to 0} \int_t^1 \frac{e^{1/x}}{\cancel{x^3}}\, \cancel{\mathrm{d}x}$$

$$= \lim_{t\to 0} \int_t^1 \frac{e^u}{x^3} \cdot -x^2\, du$$

NOTE: we need that $t \to 0^+$ (from right), otherwise $\lim_{t\to 0^-} e^{\frac{1}{t}} = 0$, because $\lim_{t\to 0^-} \frac{1}{t} = -\infty$.

$$= -\lim_{t\to 0} \int_{\frac{1}{t}}^{1} \frac{e^u}{\frac{1}{u}}\, du$$

$f(x) = u \quad g'(x) = e^u$
$f'(x) = 1 \quad g(x) = e^u$

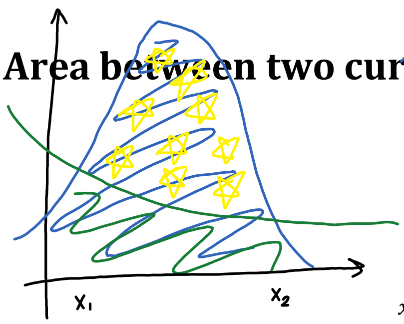
$$= -\lim_{t\to 0} \int_{\frac{1}{t}}^{1} u \cdot e^u \, du$$

$$= -\lim_{t\to 0} \left(u \cdot \mathrm{e}^u - \int_{1/t}^{1} \mathrm{e}^u \, du \right)$$

$$\vdots$$

$$= \infty$$

Area between two curves



$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x \qquad \int_{x_1}^{x_2} g(x)\,\mathrm{d}x$$

$=$ $-$

area between the curves

$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x - \int_{x_1}^{x_2} g(x)\,\mathrm{d}x = \int_{x_1}^{x_2} \big(f(x) - g(x)\big)\,\mathrm{d}x$$

upper function

lower function

Steps to finding the area between 2 curves:
1) Find intersection points between functions (equate the 2 curves).
2) Determine for sure which function is "on top".
3) Subtract functions (upper-lower).
4) Compute integral.